8/29 Recitation Solutions

1. Show that the following statements are false by finding a counterexample.

   (a) If AB and BA are defined, then A and B are square matrices.

   False. If A is $m \times n$ and B is $n \times m$, then both products are defined. So A and B need not be square matrices.

   (b) If $AB = B$ for some matrix B, then $A = I$.

   False. Let B be the zero matrix. Then $AB = 0 = B$ for any matrix A (in which AB is defined).

2. Let $A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 5 & -1 \\ -2 & 3 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 1 & -3 \\ 0 & 3 \end{bmatrix}$. Compute AB and BA, if they exist. Verify your computations using Julia.

   BA does not exist, since B has two columns while A has three rows. AB does exist, and is given by

$$AB = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 5 & -1 \\ -2 & 3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & -3 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 11 \\ 5 & -18 \\ 1 & -13 \end{bmatrix}$$

3. Prove that if A is $m \times n$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}, \mathbf{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} \in \mathbb{R}^n$, then $A(\mathbf{v} + \mathbf{w}) = A\mathbf{v} + A\mathbf{w}$.

   Hint: Use the linear combination definition of multiplication and the vector property: $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$.

   Let $\mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}, \mathbf{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$ and let A_{*i} denote the $i-$th column of A. Then

$$\begin{aligned} A(\mathbf{v} + \mathbf{w}) = A \begin{bmatrix} v_1 + w_1 \\ \vdots \\ v_n + w_n \end{bmatrix} &= (v_1 + w_1)A_{*1} + \cdots (v_n + w_n)A_{*n} \\ &= v_1 A_{*1} + \cdots + v_n A_{*n} + w_1 A_{*1} + \cdots + w_n A_{*n} \\ &= A\mathbf{v} + A\mathbf{w} \end{aligned}$$

4. Use Julia to symbolically prove the associativity property $A(BC) = (AB)C$ for some symbolic matrices A, B, and C of any (appropriate) size you choose.

Code:

```
using SymPy
m=2
n=3
p=4
q=5
A = [symbols("a$i$j") for i=1:m, j=1:n]
B = [symbols("b$i$j") for i=1:n, j=1:p]
C = [symbols("c$i$j") for i=1:p, j=1:q]
A*(B*C)==(A*B)*C
```

Output: true

5. Consider the function $T : \mathbb{R}^2 \to \mathbb{R}^3$ given by

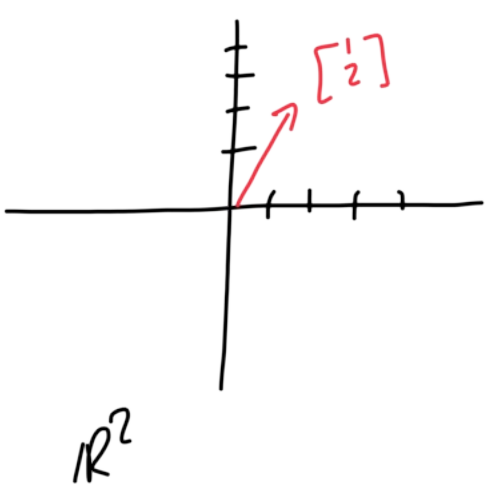
$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 3x + y \\ x - y \\ 2x \end{bmatrix}.$$

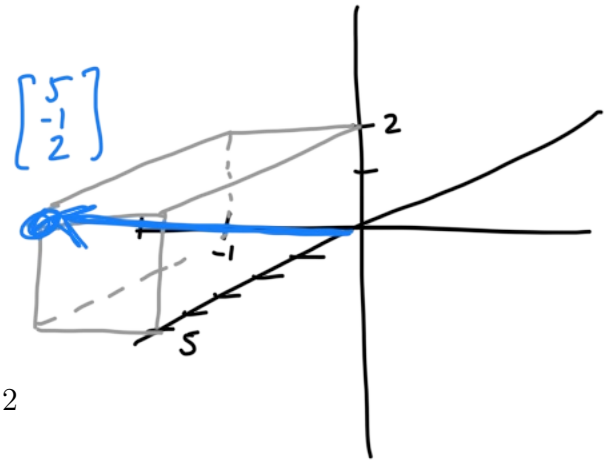
$\mathbb{R}^2$ is called the **domain** of T and $\mathbb{R}^3$ is called the **codomain** of T.

- Evaluate $T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$.

$$T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 3(1) + 2 \\ 1 - 2 \\ 2(1) \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix}$$

- Sketch the vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ in the domain of T and the vector $T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$ in the codomain of T.





6. Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T\left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} 2x - y \\ x + y \end{bmatrix}$. Find a vector $\begin{bmatrix} a \\ b \end{bmatrix}$ in the domain satisfying $T\left(\begin{bmatrix} a \\ b \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$ by reinterpreting this equation as a system of equations and solving that system.

   Since $T\left(\begin{bmatrix} a \\ b \end{bmatrix} \right) = \begin{bmatrix} 2a - b \\ a + b \end{bmatrix}$, we must solve the vector equation $\begin{bmatrix} 2a - b \\ a + b \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$ or equivalently, the system of equations

$$\begin{aligned} 2a - b &= 0 \\ a + b &= 3 \end{aligned}$$

   Adding these equations yields the equation $3a = 3$, or $a = 1$; plugging $a = 1$ back into the second equation yields $b = 2$. Hence $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.